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EXTENDED STIRLING POLYNOMIALS OF THE SECOND KIND AND EXTENDED BELL POLYNOMIALS

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ABSTRACT. Recently, several authors have studied the Stirling numbers of the second kind and Bell polynomials. In this paper, we study the extended Stirling polynomials of the second kind and the extended Bell polynomials associated with the Stirling numbers of the second kind. In addition, we note that the extended Bell polynomials can be expressed in terms of the moments of the Poisson random variable with parameter $\lambda > 0$.

1. Introduction

As is well known, the Stirling numbers of the second kind are defined as

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \ (n \geq 0), \quad (\text{see } [1-16]). \tag{1.1}$$

The generating function of $S_2(n,l)$ is given by

$$\frac{1}{m!}(e^t - 1)^m = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}, \tag{1.2}$$

where $m \in \mathbb{N} \cup \{0\}$, (see [2,7,8]).

The Stirling polynomials of the second kind are defined by the generating function

$$\frac{1}{k!}e^{xt}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k|x)\frac{t^n}{n!}, \tag{1.3}$$

where $k \geq 0$, (see [3,5,14]).

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From (1.2) and (1.3), we note that

$$S_2(n,k|x) = \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) x^{n-l} = \sum_{l=0}^{n-k} \binom{n}{l} S_2(n-l,k) x^{l}, \tag{1.4}$$

where $n, k \geq 0$, (see [3,4,5,14]).

The Bell polynomials are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x) \frac{t^n}{n!}, \quad \text{(see [7, 8, 9])}. \tag{1.5}$$

When $x = 1$, $Bel_n(1) = Bel_n$, $(n \geq 0)$, are called the Bell numbers.

From (1.2) and (1.5), we note that

$$e^{x(e^t-1)} = \sum_{m=0}^{\infty} x^m \frac{1}{m!}(e^t - 1)^m = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_2(n,m) x^m \right) \frac{t^n}{n!}. \tag{1.6}$$

Thus, by (1.6), we get

$$Bel_n(x) = \sum_{m=0}^{n} S_2(n,m) x^m, \ (n \geq 0).$$

A random variable X, taking on one of the values 0,1,2,$\cdots$, is said to be a Poisson random variable with parameter $\lambda > 0$ if $P(i) = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$, $i = 0, 1, 2, \cdots$. Note that $\sum_{i=0}^{\infty} P(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} \cdot e^{\lambda} = 1$.

The expectation of a Poisson random variable with parameter λ is given by

$$E[X] = \sum_{i=0}^{\infty} iP(i) = \sum_{i=0}^{\infty} ie^{-\lambda} \frac{\lambda^i}{i!} = \lambda. \tag{1.7}$$

The moments of Poisson random variable X with parameter $\lambda > 0$ is defined by

$$E[X^n] = \sum_{x=0}^{\infty} x^n P(x) = e^{-\lambda} \sum_{x=0}^{\infty} x^n \frac{\lambda^x}{x!}, \tag{1.8}$$

where $n \in \mathbb{N}$ (see [15]).

When $n = 1$, the first moment $E[X]$ is the mean (or expection) of X with parameter $\lambda > 0$. Recently, several authors have studied the Stirling numbers of the second kind and Bell polynomials (see [5-16]).

In this paper, we consider the extended Stirling polynomials of the second kind and the extended Bell polynomials associated with the Stirling numbers of the second kind. Then we give some identities between the extended Stirling numbers of the second kind and the extended Bell polynomials. From our new identities and properties of those numbers and polynomials, we note that the extended Bell polynomials can be expressed in terms of the moments of the Poisson random variable with parameter $\lambda > 0$.

2. Extended Stirling polynomials of the second kind and extended Bell polynomials

For $k > 0$, we define the extended Stirling polynomials of the second kind given by the generating function

$$\frac{1}{k!}e^{xt}(e^t - 1 + rt)^k = \sum_{n=k}^{\infty} S_{2,r}(n,k|x)\frac{t^n}{n!}, \tag{2.1}$$

where $x, r \in \mathbb{R}$.

When $x = 0$, $S_{2,r}(n,k|0) = S_{2,r}(n,k)$, $(n,k \geq 0)$, are called the extended Stirling numbers of the second kind. Note that $S_{2,0}(n,k) = S_2(n,k)$ are the Stirling numbers of the second kind.

It is easy to show that

$$\frac{1}{k!}e^{xt}(e^t - 1 + rt)^k = \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}\binom{n}{m}S_{2,r}(m,k)x^{n-m}\right)\frac{t^n}{n!}. \tag{2.2}$$

By (2.1) and (2.2), we get

$$S_{2,r}(n,k|x) = \sum_{m=k}^{n}\binom{n}{m}S_{2,r}(m,k)x^{n-m}, \tag{2.3}$$

where $n, k \geq 0$ and $r, x \in \mathbb{R}$.

We observe that

$$\sum_{k=0}^{\infty}\frac{1}{k!}(e^t - 1 + rt)^k = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_{2,r}(n,k)\right)\frac{t^n}{n!}, \tag{2.4}$$

and

$$\sum_{k=0}^{\infty}\frac{1}{k!}(e^t - 1 + rt)^k = e^{e^t - 1 + rt}. \tag{2.5}$$

In view of (1.5), we can define the extended Bell numbers which are given by the generating function

$$e^{e^t - 1 + rt} = \sum_{n=0}^{\infty} Bel_{n,r}\frac{t^n}{n!}, \tag{2.6}$$

From (2.4) and (2.6), we have

$$Bel_{n,r} = \sum_{k=0}^{n} S_{2,r}(n,k), \ (n \geq 0). \tag{2.7}$$

Note that $Bel_{n,0} = \sum_{k=0}^{n} S_{2,0}(n,k) = \sum_{k=0}^{n} S_2(n,k)$. Now, we define the extended Bell polynomials given by the generating function as follows:

$$e^{\lambda(e^t-1+rt)} = \sum_{n=0}^{\infty} Bel_{n,r}(\lambda)\frac{t^n}{n!}, \tag{2.8}$$

where $\lambda, r \in \mathbb{R}$.

From (2.8), we note that

$$\begin{aligned} e^{\lambda(e^t-1+rt)} &= \sum_{m=0}^{\infty} \lambda^m \frac{1}{m!}(e^t-1+rt)^m \\ &= \sum_{m=0}^{\infty} \lambda^m \sum_{n=m}^{\infty} S_{2,r}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^m S_{2,r}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.9}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$Bel_{n,r}(\lambda) = \sum_{m=0}^{n} \lambda^m S_{2,r}(n,m),$$

and

$$S_{2,r}(n,m|x) = \sum_{k=m}^{n} \binom{n}{k} S_{2,r}(k,m) x^{n-k},$$

where $n, m \geq 0$ and $r \in \mathbb{R}$.

From (1.1), we note that

$$\begin{aligned}\sum_{n=k}^{\infty} S_{2,r}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}(e^t-1+rt)^k = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}r^l t^l (e^t-1)^{k-l} \\ &= \frac{1}{k!}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}r^l t^l (e^t-1)^{k-l} \\ &= \sum_{l=0}^{k}\frac{r^l}{l!}t^l \sum_{n=k}^{\infty} S_2(n-l,k-l)\frac{t^{n-l}}{(n-l)!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{k}\binom{n}{l}r^l S_2(n-l,k-l)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Thus, by comparing the coefficients on both sides of (2.10), we obtain the following theorem.

Theorem 2.2. *For $n, k \geq 0$, we have*

$$S_{2,r}(n,k) = \sum_{l=0}^{k}\binom{n}{l}r^l S_2(n-l,k-l).$$

It is not difficult to show that

$$e^{e^t-1+rt} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}Bel_l r^{n-l}\right)\frac{t^n}{n!}. \tag{2.11}$$

Thus, by (2.6) and (2.11), we easily get

$$Bel_{n,r} = \sum_{k=0}^{n} S_{2,r}(n,k) = \sum_{k=0}^{n}\binom{n}{k}Bel_k r^{n-k}, \tag{2.12}$$

where $r \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$.

From (2.12), we have

$$Bel_{n,r} = \sum_{l=0}^{n}\binom{n}{l}Bel_{n-l}r^l = \sum_{k=0}^{n}\left(\sum_{l=0}^{k}\binom{n}{l}r^l S_2(n-l,k-l)\right).$$

By (1.2), we get

$$
\begin{aligned}
\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} &= \frac{1}{m!}(e^t-1)^m = \frac{1}{m!}(e^t-1+rt-rt)^m \\
&= \frac{1}{m!}\sum_{l=0}^{m}\binom{m}{l}t^l(e^t-1+rt)^{m-l}(-1)^l r^l \\
&= \frac{1}{m!}\sum_{l=0}^{m}\frac{m!t^l(-1)^l r^l}{l!(m-l)!}(e^t-1+rt)^{m-l} \\
&= \sum_{l=0}^{m}\frac{r^l(-1)^l}{l!}t^l\sum_{n=m-l}^{\infty}S_{2,r}(n,m-l)\frac{t^n}{n!} \\
&= \sum_{n=m}^{\infty}\left(\sum_{l=0}^{m}S_{2,r}(n-l,m-l)\binom{n}{l}(-1)^l r^l\right)\frac{t^n}{n!},
\end{aligned}
\tag{2.13}
$$

where $m \in \mathbb{N}\cup\{0\}$. By comparing the coefficients on both sides of (2.13), we obtain the following theorem.

Theorem 2.3. *For $n \geq m \geq 0$ and $r \in \mathbb{R}$, we have*

$$S_2(n,m) = \sum_{l=0}^{m}\binom{n}{l}(-1)^l r^l S_{2,r}(n-l,m-l)$$

Now, we consider the inversion formula of (2.13). From (2.1), we note that

$$
\begin{aligned}
\frac{1}{k!}(e^t-1+rt)^k &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}r^l t^l(e^t-1)^{k-l} \\
&= \frac{1}{k!}\sum_{l=0}^{k}r^l t^l\frac{k!}{l!(k-l)!}(e^t-1)^{k-l} \\
&= \sum_{l=0}^{k}\frac{r^l}{l!}t^l\sum_{n=k-l}^{\infty}S_2(n,k-l)\frac{t^n}{n!} \\
&= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{k}\binom{n}{l}r^l S_2(n-l,k-l)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.14}
$$

Therefore, by (2.1) and (2.14), we obtain the following theorem.

Theorem 2.4. *For $n \geq k \geq 0$, we have*

$$S_{2,r}(n,k) = \sum_{l=0}^{k}\binom{n}{l}r^l S_2(n-l,k-l).$$

For $m,n,k\in\mathbb{N}$, we have

$$\begin{aligned}&\frac{1}{m!}(e^t-1+rt)^m\frac{1}{k!}(e^t-1+rt)^k=\frac{1}{m!k!}(e^t-1+rt)^{m+k}\\&=\frac{(m+k)!}{m!k!}\frac{1}{(m+k)!}(e^t-1+rt)^{m+k}=\binom{m+k}{m}\sum_{n=m+k}^{\infty}S_{2,r}(n,m+k)\frac{t^n}{n!}.\end{aligned}\tag{2.15}$$

On the other hand,

$$\begin{aligned}&\frac{1}{m!}(e^t-1+rt)^m\frac{1}{k!}(e^t-1+rt)^k\\&=\left(\sum_{l=m}^{\infty}S_{2,r}(l,m)\frac{t^l}{l!}\right)\left(\sum_{j=k}^{\infty}S_{2,r}(j,k)\frac{t^j}{j!}\right)\\&=\sum_{n=k+m}^{\infty}\left(\sum_{l=m}^{n}\frac{n!}{l!(n-l)!}S_{2,r}(l,m)S_{2,r}(n-l,k)\right)\frac{t^n}{n!}\\&=\sum_{n=k+m}^{\infty}\left(\sum_{l=m}^{n}\binom{n}{l}S_{2,r}(l,m)S_{2,r}(n-l,k)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.16}$$

Therefore, by (2.15) and (2.16), we obtain the following theorem.

Theorem 2.5. *For $n,m,k\geq 0$ with $n\geq m+k$, we have*

$$\binom{m+k}{m}S_{2,r}(n,m+k)=\sum_{l=m}^{n}\binom{n}{l}S_{2,r}(l,m)S_{2,r}(n-l,k).$$

Now, we observe that

$$\begin{aligned}&\frac{1}{m!}(e^t-1+rt)^m\frac{1}{k!}(e^t-1+rt)^k\\&=\left(\frac{1}{m!}\sum_{l=0}^{m}\binom{m}{l}(e^t-1)^{m-l}r^lt^l\right)\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(e^t-1)^{k-j}r^jt^j\right)\\&=\left(\sum_{l=0}^{m}\frac{r^lt^l}{l!}\sum_{n_1=m}^{\infty}S_2(n_1-l,m-l)\frac{t^{n_1-l}}{(n_1-l)!}\right)\\&\quad\times\left(\sum_{j=0}^{k}\frac{r^jt^j}{j!}\sum_{n_2=k}^{\infty}S_2(n_2-j,k-j)\frac{t^{n_2-j}}{(n_2-j)!}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{n_1=m}^{\infty} \left(\sum_{l=0}^{m} \binom{n_1}{l} r^l S_2(n_1-l, m-l) \right) \frac{t^{n_1}}{n_1!} \right) \\
&\quad \times \left(\sum_{n_2=k}^{\infty} \left(\sum_{j=0}^{k} \binom{n_2}{j} r^j S_2(n_2-j, k-j) \right) \frac{t^{n_2}}{n_2!} \right) \\
&= \sum_{n=m+k}^{\infty} \left\{ \sum_{n_1=m}^{n} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{n_1}{l} \binom{n-n_1}{j} r^{l+j} \binom{n}{n_1} \right. \\
&\quad \left. \times S_2(n_1-l, m-l) S_2(n-n_1-j, k-j) \right\} \frac{t^n}{n!}.
\end{aligned} \tag{2.17}
$$

By (2.15) and (2.17), we get

$$
\begin{aligned}
&\binom{m+k}{m} S_{2,r}(n, m+k) \\
&= \sum_{n_1=m}^{n} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{n_1}{l} \binom{n-n_1}{j} \binom{n}{n_1} r^{l+j} S_2(n_1-l, m-l) S_2(n-n_1-j, k-j).
\end{aligned} \tag{2.18}
$$

With $r = 0$ in (2.15), we have

$$
\frac{1}{m!}(e^t-1)^m \frac{1}{k!}(e^t-1)^k = \binom{k+m}{m} \sum_{n=k+m}^{\infty} S_2(n, m+k) \frac{t^n}{n!}, \tag{2.19}
$$

where $n, m, k \geq 0$. From (1.2) and (2.1), we have

$$
\begin{aligned}
&\frac{1}{m!}(e^t-1)^m \frac{1}{k!}(e^t-1)^k = \frac{1}{m!}(e^t-1+rt-rt)^m \frac{1}{k!}(e^t-1+rt-rt)^k \\
&= \left(\frac{1}{m!} \sum_{l=0}^{m} \binom{m}{l} (e^t-1+rt)^{m-l}(-rt)^l \right) \left(\frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (e^t-1+rt)^{k-j}(-rt)^j \right) \\
&= \left(\sum_{l=0}^{m} \frac{(-1)^l r^l}{l!} t^l \sum_{n_1=m}^{\infty} S_{2,r}(n_1-l, m-l) \frac{t^{n_1-l}}{(n_1-l)!} \right) \\
&\quad \times \left(\sum_{j=0}^{k} \frac{(-1)^j r^j}{j!} t^j \sum_{n_2=k}^{\infty} S_{2,r}(n_2-j, k-j) \frac{t^{n_2-j}}{(n_2-j)!} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{n_1=m}^{\infty} \sum_{l=0}^{m} \binom{n_1}{l} (-1)^l r^l S_{2,r}(n_1 - l, m - l) \frac{t^{n_1}}{n_1!} \right) \\
&\quad \times \left(\sum_{n_2=k}^{\infty} \sum_{j=0}^{k} \binom{n_2}{j} (-1)^j r^j S_{2,r}(n_2 - j, k - j) \frac{t^{n_2}}{n_2!} \right) \\
&= \sum_{n=m+k}^{\infty} \left\{ \sum_{n_1=m}^{n} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{n_1}{l} \binom{n-n_1}{j} \binom{n}{n_1} (-1)^{l+j} r^{l+j} \right. \\
&\qquad \left. \times S_{2,r}(n_1 - l, m - l) S_{2,r}(n - n_1 - j, k - j) \right\} \frac{t^n}{n!}.
\end{aligned} \tag{2.20}
$$

Comparing the coefficients on both sides of (2.19) and (2.20), we have

$$
\begin{aligned}
&\binom{k+m}{m} S_2(n, m+k) \\
&= \sum_{n_1=m}^{n} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{n_1}{l} \binom{n-n_1}{j} \binom{n}{n_1} (-1)^{l+j} r^{l+j} \\
&\quad \times S_{2,r}(n_1 - l, m - l) S_{2,r}(n - n_1 - j, k - j),
\end{aligned} \tag{2.21}
$$

where $n, m, k \geq 0$ with $n \geq m + k$.

3. Further Remarks

A random variable X, taking on one of the values $0, 1, 2, \cdots$, is said to be a Poisson random variable with parameter $\lambda > 0$ if $P(i) = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$, $i = 0, 1, 2, \cdots$. Note that $\sum_{i=0}^{\infty} P(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$.

The Bell polynomials $Bel_n(x)$, $(n \geq 0)$, are known to be connected with the Poisson distribution. More precisely, $Bel_n(\lambda)$ can be expressed in terms of the moments of Poisson random variable x with parameter $\lambda > 0$ as

$$Bel_n(\lambda) = E[X^n], \ (n \in \mathbb{N}).$$

Let X be a Poisson random variable with paramerer $\lambda > 0$. Then we observe that

$$
\begin{aligned}
E[e^{t(X+r\lambda)}] &= \sum_{n=0}^{\infty} E[(X+r\lambda)^n] \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{x=0}^{\infty} (x+r\lambda)^n \frac{\lambda^x}{x!} e^{-\lambda} \right) \frac{t^n}{n!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \left(\sum_{n=0}^{\infty} (x+r\lambda)^n \frac{t^n}{n!} \right) \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} e^{(x+r\lambda)t} \frac{\lambda^x}{x!} = e^{rt\lambda-\lambda} \sum_{x=0}^{\infty} e^{xt} \frac{\lambda^x}{x!} \\
&= e^{\lambda(e^t-1+rt)} = \sum_{n=0}^{\infty} Bel_{n,r}(\lambda) \frac{t^n}{n!}.
\end{aligned} \tag{3.1}
$$

Thus, by (3.1), we see that the extended Bell polynomials are expressed in terms of the moments of Poisson random variable X with parameter $\lambda > 0$ as follows:

$$
E[(X+r\lambda)^n] = Bel_{n,r}(\lambda), \tag{3.2}
$$

where $n \in \mathbb{N}$ and $r \in \mathbb{R}$. By binomial theorem, we get

$$
(X+r\lambda)^n = \sum_{l=0}^{n} \binom{n}{l} r^l \lambda^l X^{n-l}. \tag{3.3}
$$

Thus, by (3.3), we get

$$
\begin{aligned}
Bel_{n,r}(\lambda) = E[(X+r\lambda)^n] &= \sum_{l=0}^{n} \binom{n}{l} r^l \lambda^l E[X^{n-l}] \\
&= \sum_{l=0}^{n} \binom{n}{l} r^l \lambda^l Bel_{n-l}(\lambda).
\end{aligned} \tag{3.4}
$$

From (2.1) and (3.1), we note that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} E[(X+r\lambda)^n] \frac{t^n}{n!} = e^{\lambda(e^t-1+rt)} \\
&= \sum_{m=0}^{\infty} \lambda^m \frac{1}{m!} (e^t - 1 + rt)^m = \sum_{m=0}^{\infty} \lambda^m \sum_{n=m}^{\infty} S_{2,r}(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^m S_{2,r}(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.5}
$$

Thus, by comparing the coefficients on both sides of (3.5), we get

$$E[(X+r\lambda)^n] = \sum_{m=0}^{n} \lambda^m S_{2,r}(n,m) = Bel_{n,r}(\lambda), \tag{3.6}$$

where $n \in \mathbb{N} \cup \{0\}$ and X is a Poisson random variable with parameter $\lambda > 0$.

Now, we observe that

$$\begin{aligned} & e^{tx} E[e^{t(X+r\lambda)}] \\ &= \left(\sum_{l=0}^{\infty} \frac{x^l}{l!} t^l\right) \left(\sum_{m=0}^{\infty} E[(X+r\lambda)^m] \frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} x^{n-m} E[(X+r\lambda)^m]\right) \frac{t^n}{n!}. \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned} & e^{tx} E[e^{t(X+r\lambda)}] = e^{\lambda(e^t-1+rt)} e^{xt} \\ &= \sum_{k=0}^{\infty} \lambda^k \frac{1}{k!} (e^t - 1 + rt)^k e^{xt} \\ &= \sum_{k=0}^{\infty} \lambda^k \sum_{n=k}^{\infty} S_{2,r}(n,k|x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \lambda^k S_{2,r}(n,k|x)\right) \frac{t^n}{n!}. \end{aligned} \tag{3.8}$$

Thus, by (3.7) and (3.8), we get

$$\sum_{m=0}^{n} \binom{n}{m} x^{n-m} E[(X+r\lambda)^m] = \sum_{k=0}^{n} \lambda^k S_{2,r}(n,k|x), \tag{3.9}$$

where $n, k \geq 0$ and X is Poisson random variable with parameter $\lambda > 0$.

The (3.9) is equivalent to

$$\begin{aligned} & \sum_{m=0}^{n} \binom{n}{m} x^{n-m} Bel_{m,r}(\lambda) \\ &= \sum_{k=0}^{n} \lambda^k S_{2,r}(n,k|x), \text{ where } n \geq 0,\ r \in \mathbb{R}. \end{aligned} \tag{3.10}$$

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